Regular games: A playground for exponential time algorithms

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Plan

- Regular games: basic definitions
- The sizes and parameters of the games
- Our results on
  - Coloured Muller games
  - Rabin and Streett games
  - Muller games
  - McNaughton games

For the paper see Archives:
Deciding regular games:
A playground for exponential time algorithms

Arenas and plays

Definition

An **arena** $\mathcal{A}$ is a bipartite directed graph (V_0, V_1, E), where

1. $V_0 \cap V_1 = \emptyset$, and $V = V_0 \cup V_1$ is the set of **positions.**
2. $E \subseteq V_0 \times V_1 \cup V_1 \times V_0$ is the edge set where each node has an outgoing edge.
3. V_0 and V_1 are positions for Player 0 and Player 1.

Plays

Given a token initially placed on a position v, players move the token in turn along the edges. This produces a path.

Definition

Let $\mathcal{A}$ be an arena. A **play** starting at $v_0 \in V$, is an infinite sequence $\rho = v_0, v_1, v_2, \ldots$ such that $v_{i+1} \in E(v_i)$ for all $i \in \mathbb{N}$.

Recurring states in plays

Given a play $\rho = v_0, v_1, \ldots$, the set

$$\mathrm{Inf}(\rho) = \{v \in V \mid \exists^{\omega} i(v_i = v)\}$$

is called the **infinity set** of ρ.

The winner of ρ is determined by a condition put on $\mathrm{Inf}(\rho)$.

We list several well-established winning conditions.

- A **coloured Muller game** is $\mathcal{G} = (\mathcal{A}, c, (\mathcal{F}_0, \mathcal{F}_1))$, where $c : V \to C$, $\mathcal{F}_0 \cup \mathcal{F}_1 = 2^C$ and $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$. The sets $\mathcal{F}_0$ and $\mathcal{F}_1$ are **winning conditions**. Player σ **wins** the play ρ if $c(\mathrm{Inf}(\rho)) \in \mathcal{F}_\sigma$, where $\sigma = 0, 1$.

- A **Rabin game** is the tuple $\mathcal{G} = (\mathcal{A}, (U_1, V_1), \ldots, (U_k, V_k))$, where $U_i, V_i \subseteq V$, (U_i, V_i) is a **winning pair**, and k is the **index**. Player 0 **wins** ρ if there is a pair (U_i, V_i) with $\mathrm{Inf}(\rho) \cap U_i \neq \emptyset$ and $\mathrm{Inf}(\rho) \cap V_i = \emptyset$. Else, Player 1 wins.

- A **Streett game** is the tuple $\mathcal{G} = (\mathcal{A}, (U_1, V_1), \ldots, (U_k, V_k))$, where U_i, V_i are as in Rabin game. Player 0 **wins** ρ if for all $i \in \{1, \ldots, k\}$ if $\mathrm{Inf}(\rho) \cap U_i \neq \emptyset$ then $\mathrm{Inf}(\rho) \cap V_i \neq \emptyset$. Otherwise, Player 1 wins.

McNaughoton and Muller conditions

- A **McNaughton game** is the tuple $\mathcal{G} = (\mathcal{A}, W, (\mathcal{F}_0, \mathcal{F}_1))$, where $W \subseteq V$, $\mathcal{F}_0 \cup \mathcal{F}_1 = 2^W$ and $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$. Player σ **wins** the play ρ if $\mathrm{Inf}(\rho) \cap W \in \mathcal{F}_\sigma$.
- A **Muller game** is the tuple $\mathcal{G} = (\mathcal{A}, (\mathcal{F}_0, \mathcal{F}_1))$, where $\mathcal{F}_0 \cup \mathcal{F}_1 = 2^V$ and $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$. Player σ **wins** the play ρ if $\mathrm{Inf}(\rho) \in \mathcal{F}_\sigma$.

Sizes and parameters of games

There are two ways to compute the size of $\mathcal{G}$:

1. The size of $\mathcal{G}$ is $|V| + |E|$.
2. The size of $\mathcal{G}$ is $|V| + |E| +$ |Winning condition|:
    1. The sizes of Muller, McNaughton, and coloured Muller games are bounded by $|V| + |E| + 2^{|V|} \cdot |V|$.
    2. The sizes of Rabin and Streett games are bounded by $|V| + |E| + 4^{|V|} \cdot |V|$.

The *small* parameters are $|C|$ and $|W|$.
The *large* parameter is k.

Solving regular games

Theorem (Determinacy Theorem, folklore)

The set of positions of the game $\mathcal{G}$ can be partitioned into W_0 and W_1 such that:

1. $x \in W_0$ *iff Player 0 wins* $\mathcal{G}$ *starting at* x.
2. $y \in W_1$ *iff Player 1 wins* $\mathcal{G}$ *starting at* y.

Solving a game has two objectives:

Objective 1. Given $\mathcal{G}$ compute W_0 and W_1.
Let us call this the **decision problem.**

Objective 2. Extract winning strategies for the winners.

We focus on the decision problem.

Our results: colored Muller games

A **coloured Muller game** is $\mathcal{G} = (\mathcal{A}, c, (\mathcal{F}_0, \mathcal{F}_1))$, where $c : V \to C$, $\mathcal{F}_0 \cup \mathcal{F}_1 = 2^C$ and $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$. Player σ **wins** the play ρ if $c(\mathrm{Inf}(\rho)) \in \mathcal{F}_\sigma$.

Best known (running time, space):

- $(O(|C|^{5|C|} \cdot |V|^5), O((|C|!|V|)^{O(1)}))$ (STOC 2017, Calude, Jain, Khoussainov, Stephan, Li)
- $(O(|C||E|(|C||V|)^{|C|-1}), O(|\mathcal{G}| + |C||V|))$ (folklore)

Our results (running time, space)

- $(O(2^{|V|}|C||E|), O(|\mathcal{G}| + 2^{|V|}|V|))$ (DP)
- $(O(2^{|V|}|V||E|), O(|\mathcal{G}| + 2^{|V|}))$ (DP)
- $(O(|C|!\binom{|V|}{|C|}|V||E|), O(|\mathcal{G}| + |C||V|))$ (Recursion)

Comments:

If $|V|/\log\log(n) \leq |C|$, then for the DP algorithms we have:

1. Run times are better than $O(|C|^{5|C|} \cdot |V|^5)$.
2. The running times strengthen the impossibility result that under the ETH colored Muller games cannot be in $2^{o(|C| \cdot \log(|C|))} Poly(|V|)$.
3. The spaces are better than $O((|C|!|V|)^{O(1)})$.
4. All of the previous algorithms run in superexponential times. Our algorithms are in EXP.

Example

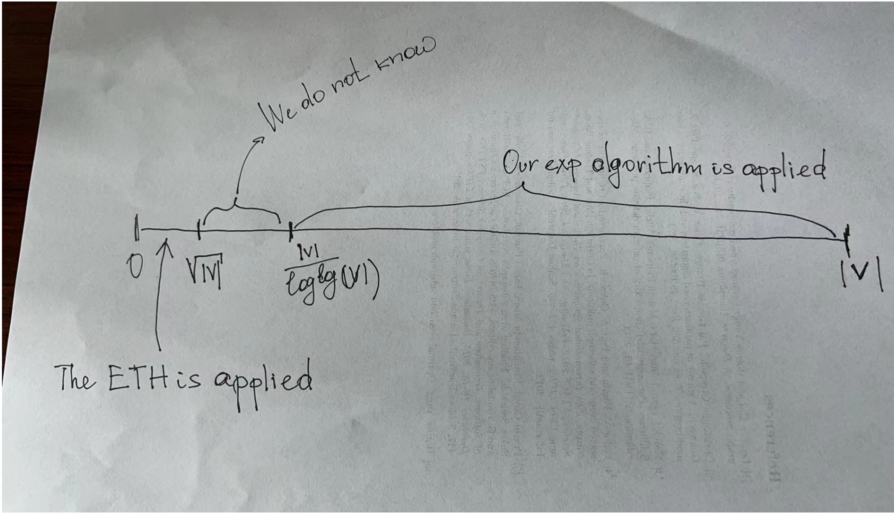

Figure: The values of $|C|$ and the ETH

Our results: Rabin and Streett games

A **Rabin game** is the tuple $\mathcal{G} = (\mathcal{A}, (U_1, V_1), \ldots, (U_k, V_k))$. Player 0 **wins** ρ if there is a pair (U_i, V_i) with $\mathrm{Inf}(\rho) \cap U_i \neq \emptyset$ and $\mathrm{Inf}(\rho) \cap V_i = \emptyset$. Else, Player 1 wins.

Best known (running time, space):

- $(O(|E||V|^{k+1}kk!), O(|\mathcal{G}| + k|V|))$
  (N. Piterman and A. Pnuelli LICS 2006)
- $(\tilde{O}(|E||V|(k!)^{1+o(1)}), O(|\mathcal{G}| + k|V| \log k \log |V|))$
  (R. Majumdar et al. 2024)

Our results (running time, space):

- $(O((k|V| + 2^{|V|}|E|)|V|), O(|\mathcal{G}| + 2^{|V|}|V|)$ (DP)
- $(O(|V|!|V|(|E| + k|V|)), O(|\mathcal{G}| + |V|^2))$ (Recursion)

Comments:

1. In terms of time, both our DP and recursive algorithms are better when $k \in [|V|, 4^{|V|}]$.
2. We refine the impossibility result of A. Casares et al. (SOSA 2024) under the assumption of the ETH. When $k \geq |V| \log |V|$, both algorithms run in $2^{o(k \log k)} Poly(|V|)$.
3. Our DP algorithm is the first exponential time algorithm that decides Rabin games.
4. When $k \in [|V|, 4^{|V|}]$, then the recursive algorithm performs the best in terms of space against other algorithms.

Our results: Muller games

A **Muller game** is the tuple $\mathcal{G} = (\mathcal{A}, (\mathcal{F}_0, \mathcal{F}_1))$, where $\mathcal{F}_0 \cup \mathcal{F}_1 = 2^V$ and $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$. Player σ **wins** ρ if $\mathrm{Inf}(\rho) \in \mathcal{F}_\sigma$.

Best known (running time, space):

- $(O(|\mathcal{F}_0| \cdot (|V| + |\mathcal{F}_0|) \cdot |V_0| \log |V_0|), O(|\mathcal{G}| + |\mathcal{F}_0|(|V| + |\mathcal{F}_0|)))$ (B. Khoussainov, Z. Liang, and M. Xiao ESA 2023)

Our results (running time, space):

- $(O(2^{|V|}|V||E|), O(|\mathcal{G}| + 2^{|V|}))$

Our algorithm becomes competitive (or better) than the state of the art when $|\mathcal{F}_0| > \sqrt{2^{|V|}}$.

Our results: McNaughton games

A **McNaughton game** is the tuple $\mathcal{G} = (\mathcal{A}, W, (\mathcal{F}_0, \mathcal{F}_1))$, where $W \subseteq V$, $\mathcal{F}_0 \cup \mathcal{F}_1 = 2^W$ and $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$

Best known (running time, space):

- $(O(|W||E||W|!), O(|\mathcal{G}| + Poly(|V|)))$
  (folklore, e.g., see
  McNaughton APAL 1993, Nerode et al APAL 1996)

Our results (running time, space):

- $(O(2^{|V|}|W||E|), O(|\mathcal{G}| + 2^{|V|}|V|))$ (DP)
- $(O(2^{|V|}|V||E|), O(|\mathcal{G}| + 2^{|V|}))$ (DP)

When $|W| \geq |V|/\log\log(n)$, then our algorithm has asymptotically better running time.

Why do we have such improvements?

1. Most algorithms involve parameters such as C, W, and k. These connections make the algorithms complex.
2. When $|C|$, $|W|$, and k move from very small to reasonable, the run times of the algorithms become unreasonable (that involve $|C|^{|C|}$, $|C|!$, $|W|!$, $(k!)^{1+o(1)}$, $|V|^k$, etc).
3. Hence, it is better to run algorithms that go through all subsets of the arena resulting in exponential bounds.
4. Even when one runs through the subsets of the arenas, a non-trivial technical work needs to be done.
   This is explained in the next few slides.

Full win

Definition

If $Win_\sigma(\mathcal{G}) = V$, then player σ **fully wins** $\mathcal{G}$. Else, the player **does not fully win** $\mathcal{G}$. If $Win_\sigma(\mathcal{G}) \neq V$ and $Win_{\bar{\sigma}}(\mathcal{G}) \neq V$, then **no player fully wins** $\mathcal{G}$.

Attractors

Let $X \subset V$ and let us fix Player σ.

The attractor of X for Player σ is the collection of all positions x from which Player σ can force the token into X.

We denote this set of positions by

$$Attr_{\sigma}(X, \mathcal{A}).$$

Full win lemma for colored Muller games

Lemma

Let $\sigma \in \{0,1\}$ such that $c(V) \in \mathcal{F}_\sigma$. Then:

1. *If for all $c' \in c(V)$, $Attr_\sigma(c^{-1}(c'), \mathcal{A}) = V$ or Player σ fully wins $\mathcal{G}(V \setminus Attr_\sigma(c^{-1}(c'), \mathcal{A}))$, then Player σ fully wins $\mathcal{G}$.*
2. *Else, let $c' \in C$ be such that $Attr_\sigma(c^{-1}(c'), \mathcal{A}) \neq V$ and Player σ doesn't fully win $\mathcal{G}(V \setminus Attr_\sigma(c^{-1}(c'), \mathcal{A}))$. Then $Win_\sigma(\mathcal{G}) = Win_\sigma(\mathcal{G}(V \setminus X))$, where $X = Attr_{\bar{\sigma}}(Win_{\bar{\sigma}}(\mathcal{G}(V \setminus Attr_\sigma(c^{-1}(c'), \mathcal{A}))), \mathcal{A})$.*

Corollary

Assume that $c(V) \in \mathcal{F}_\sigma$. Player σ fully wins $\mathcal{G}$ iff for all $c' \in c(V)$, $Attr_\sigma(c^{-1}(c'), \mathcal{A}) = V$ or Player σ fully wins $\mathcal{G}(V \setminus Attr_\sigma(c^{-1}(c'), \mathcal{A}))$. □

Trichotomy Lemma for colored Muller game

Lemma (Trichotomy Lemma)

Let $\sigma \in \{0, 1\}$ be such that $c(V) \in \mathcal{F}_\sigma$. Then:

1. *If for all $c' \in c(V)$, $Attr_\sigma(c^{-1}(c'), \mathcal{A}) = V$ or Player σ fully wins $\mathcal{G}(V \setminus Attr_\sigma(c^{-1}(c'), \mathcal{A}))$, then Player σ fully wins $\mathcal{G}$.*
2. *Otherwise, if for all $v \in V$, $Attr_{\bar{\sigma}}(\{v\}, \mathcal{A}) = V$ or Player $\bar{\sigma}$ fully wins $\mathcal{G}(V \setminus Attr_{\bar{\sigma}}(\{v\}, \mathcal{A}))$, then Player $\bar{\sigma}$ fully wins $\mathcal{G}$.*
3. *Otherwise, none of the players fully wins.*

This lemma is used for our dynamic programming algorithms.

Enumeration Lemma

Lemma (Enumeration Lemma)

Given the set $\mathcal{S} = \{V_1, \ldots, V_k\}$ be subsets of V, where $n = |V|$. Then the collection

$$2^{V_1} \cup \ldots \cup 2^{V_k}$$

can be enumerated in time $O(2^n n)$ and space $O(2^n)$.

The obvious algorithm runs in $O(2^n k)$. The lemma removes dependence on k.

KL winning condition

- A **KL game** is the tuple $\mathcal{G} = (\mathcal{A}, (u_1, S_1), \ldots, (u_t, S_t))$, where $u_i \in V$, $S_i \subseteq V$, (u_i, S_i) is a **winning pair**, and t is the **index**. Player 0 **wins** ρ if there is a pair (u_i, S_i) such that $u_i \in \mathrm{Inf}(\rho)$ and $\mathrm{Inf}(\rho) \subseteq S_i$. Else, Player 1 wins.

The size of the games with KL condition is bounded by $|V| + |E| + 2^{|V|} \cdot |V|^2$.

From KL condition to Muller condition

Theorem

There is a transformation from KL games $\mathcal{G}$ to Muller games $\mathcal{G}'$ that takes $O(2^{|V|}|V|^2)$ time and $O(|\mathcal{G}| + 2^{|V|})$ space. Hence, there exists an algorithm that, given a KL game $\mathcal{G}$, decides $\mathcal{G}$ in $O(2^{|V|}|V||E|)$ time and $O(|\mathcal{G}| + 2^{|V|})$ space.

From Rabin to KL condition

Theorem

There is a transformation from Rabin games $\mathcal{G}$ to KL games that takes time $O(k|V|^2)$ and space $O(|\mathcal{G}| + 2^{|V|}|V|)$. Hence, there exist algorithms that decide Rabin and Streett games $\mathcal{G}$ in $O((k|V| + 2^{|V|}|E|)|V|)$ time and $O(|\mathcal{G}| + 2^{|V|}|V|)$ space.

Open questions

1. Muller games can be decided in Polynomial time. Can we decide McNaughton games in polynomial time?

2. Are there exponential time algorithms that decide coloured Muller games when the parameter $|C|$ ranges in the interval $[\sqrt{|V|}, |V|/a]$, where $a > 1$.

3. Can we replace the factor $2^{|V|}$ with $2^{|W|}$ in the running time that decides McNaughton games? If this can be done, then one implies that the ETH is not applicable to McNaughton games as opposed to coloured Muller games and Rabin games.